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Recent Developments in the Analysis of Coupled Oscillator Arrays

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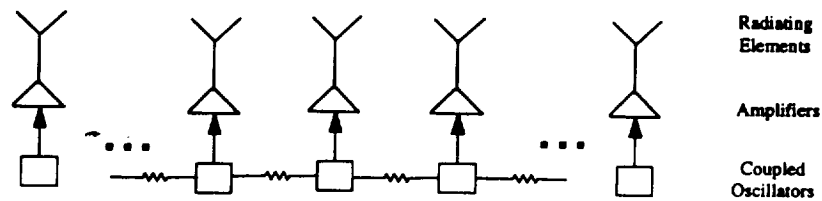
Introduction

- Consider a linear array of coupled oscillators.
 - Achieves high radiated power through coherent spatial power combining.
 - Usually designed to produce constant aperture phase.
- Oscillators are injection locked to each other or to a master oscillator to produce coherent radiation.
- Oscillators do not necessarily oscillate at their tuning frequency.
- York, et. al. have shown that the phase of each oscillator is a function of the difference between the *tuning frequency* and the *oscillation frequency*.

2

Our purpose in coupling oscillators together is to achieve high radiated power through the spatial power combining which results when the oscillators are injection locked to each other. York, et. al. have shown that, left to themselves, the ensemble of injection locked oscillators oscillate at the average of the tuning frequencies of all the oscillators.

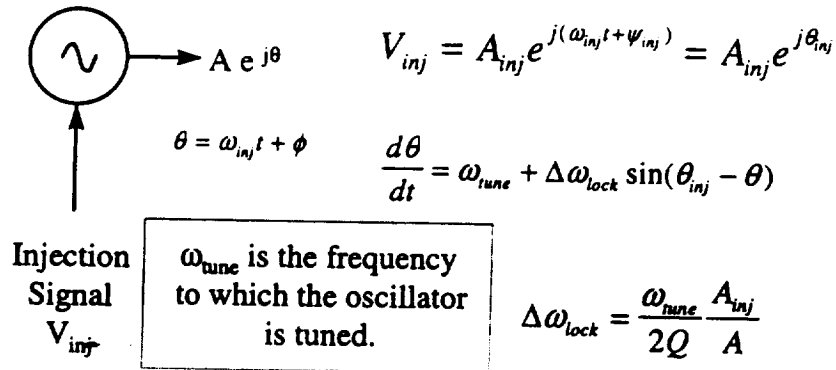
Coupled Oscillators for Radiating Aperture Phase Control



3

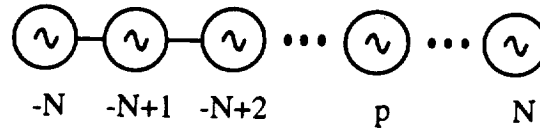
This shows the concept of controlling the phase in a radiating aperture using coupled electronic oscillators. The amplifiers serve two purposes; they provide for high radiated power and they isolate the oscillators from the parasitic coupling between the radiating elements thus permitting more precise control of the nature of the interoscillator coupling.

Injection Locking



Consider a single injection locked oscillator. We represent the signals as complex functions as indicated. In steady state, of course, the oscillator will oscillate at the injection frequency. The transient (time varying) behavior is governed by the indicated differential equation. Using this equation we can formulate the theory of a set of coupled oscillators.

Coupled Oscillators



$$\frac{d\theta_i}{dt} = \omega_{\text{tune}} - \frac{\omega}{2Q} \sum_{j=i-1}^{j=i+1} \epsilon_{ij} \frac{A_j}{A_i} \sin(\Phi_{ij} + \theta_i - \theta_j)$$

In the continuum model:

$$\nabla^2 \theta - \frac{\partial \theta}{\partial \tau} = - \frac{\omega_{\text{tune}}}{\Delta \omega_{\text{lock}}}$$

5

Here we adapt the preceding differential equation to describe the behavior of a linear array of coupled oscillators with nearest neighbor coupling. Using a continuum model of this description leads to the partial differential equation shown at the bottom of the vugraph. Tau is time multiplied by the locking bandwidth of the oscillators.

Coupled Oscillators (Continued)

Define the phase of the i th oscillator, ϕ_i , by:

$$\theta_i = \omega_{ref} t + \phi_i$$

Then,

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial \phi}{\partial \tau} = -\frac{\omega_{tune} - \omega_{ref}}{\Delta \omega_{lock}} = -Cu(\tau)\delta(x - b)$$

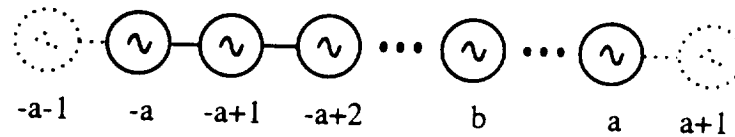
Laplace transformation yields,
$$\frac{\partial^2 f}{\partial x^2} - sf = -\frac{C}{s}\delta(x - b)$$

6

We define the phase of the i th oscillator with respect to a reference frequency to be selected to be the initial ensemble frequency of the array which has been shown to be the initial average of the oscillator tuning frequencies. In the case where the oscillator at $x=b$ is detuned by C (measured in locking ranges), the partial differential equation we wish to solve take the form shown.

Boundary Conditions

Boundary conditions can be derived from,



Tune added oscillators so that,

$$\phi(-a-1) = \phi(-a)$$

Then,

$$\left. \frac{\partial \phi}{\partial x} \right|_{-a-\frac{1}{2}} = 0$$

$$\phi(a+1) = \phi(a)$$

$$\left. \frac{\partial \phi}{\partial x} \right|_{a+\frac{1}{2}} = 0$$

**That is, the classical
Neumann conditions.**

7

To find the solution corresponding to an array of finite length, $2a$, one must effectively add homogeneous solutions of the equation to the particular integral in sufficient amounts to satisfy the boundary conditions at the ends of the array. These boundary conditions can be ascertained using the artifice indicated here. That is, two fictitious oscillators are added to the array, one at each end. These oscillators are assumed to be dynamically tuned in such a manner as to maintain their phase equal to the phase of the corresponding actual end oscillator. This condition assures that no injection effect is transmitted between these pairs of oscillators. This shows that the correct boundary condition is one of classical Neumann type applied one half unit cell outside each end of the array.

The Finite Array Solution

The solution can be immediately written as a sum of the eigenfunctions.

$$\phi(x, s) = -\frac{C}{s} \sum_{n=0}^{\infty} \frac{2 \cosh(b\sqrt{s_n}) \cosh(x\sqrt{s_n})}{(2a+1)(s_n - s)} + \frac{C}{s} \sum_{m=0}^{\infty} \frac{2 \sinh(b\sqrt{s_m}) \sinh(x\sqrt{s_m})}{(2a+1)(s_m - s)}$$

and the inverse Laplace transform is merely the sum of the residues; i.e.,

$$\begin{aligned} \phi(x, \tau) = & \frac{C\tau}{2a+1} + C \sum_{n=1}^{\infty} \frac{2 \cos(b\sqrt{\sigma_n}) \cos(x\sqrt{\sigma_n})}{(2a+1)\sigma_n} + C \sum_{m=0}^{\infty} \frac{2 \sin(b\sqrt{\sigma_m}) \sin(x\sqrt{\sigma_m})}{(2a+1)\sigma_m} \\ & - C \sum_{n=0}^{\infty} \frac{2 \cos(b\sqrt{\sigma_n}) \cos(x\sqrt{\sigma_n})}{(2a+1)\sigma_n} e^{-\sigma_n \tau} - C \sum_{m=0}^{\infty} \frac{2 \sin(b\sqrt{\sigma_m}) \sin(x\sqrt{\sigma_m})}{(2a+1)\sigma_m} e^{-\sigma_m \tau} \end{aligned}$$

8

The solution can be written using the well established theoretical foundation of Sturm and Liouville. The inverse Laplace transform follows immediately in view of the simple pole appearing in each term of the sum. There is a simple pole at the origin resulting from the denominator s in front of the summations and this leads to the steady state solution as an eigenfunction sum. In addition, there is one double pole at the origin leading to the term linear in time. This term is a manifestation of the shift of the array ensemble frequency from that before the step detuning of the oscillator at $x=b$.

The Finite Array (Continued)

It should be noted that the time constants in the dynamic solution are given by the eigenvalues. The slowest time constants are:

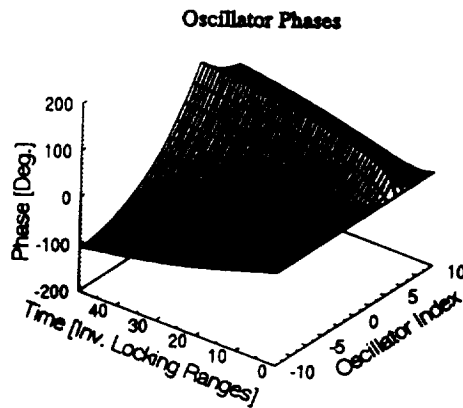
$$\sigma_{\min} \approx \left(\frac{\pi}{2a+1} \right)^2 \quad \text{for nonsymmetrical detuning}$$

$$\sigma_{\min} \approx \left(\frac{2\pi}{2a+1} \right)^2 \quad \text{for symmetrical detuning}$$

In both cases the response time is proportional to the *square* of the number of oscillators in the array.

A significant result of this analysis is that the response time of such an array is proportional to the square of the number of oscillators. This is apparently a well known result in diffusion theory and arises here because the differential equation governing the phase dynamics is of diffusion type.

The Finite Array Solution



10

This is a graphical representation of the solution for the finite length array with oscillator "5" step detuned at $t=0$. Here again it is merely a plot of the analytical solution obtained via the Laplace transformation.

Beamsteering Dynamics

Equal and opposite detuning of the end oscillators; i.e.,

$$\Delta\omega_L = -\Delta\omega_R = \Delta\omega_T$$

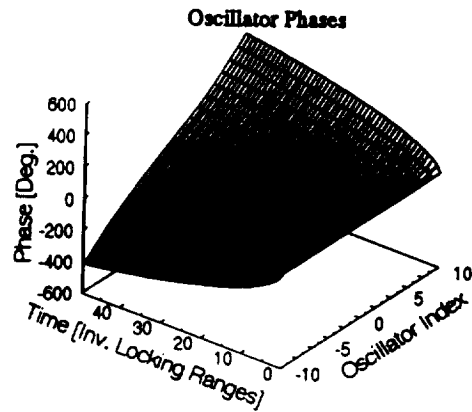
yields,

$$\tilde{\phi}(x, \tau) = \frac{\Delta\omega_T}{\Delta\omega_{lock}} \sum_{m=0}^{\infty} \frac{2 \sin(b\sqrt{\sigma_m}) \sin(x\sqrt{\sigma_m})}{(2a+1)\sigma_m} (1 - e^{-\sigma_m \tau})$$

11

According to Liao, et.al. [IEEE Trans. MTT-41, pp. 1810-18115, Oct. 1993], beamsteering is accomplished by equal and opposite detuning of the end oscillators of the array. The solution for the phase distribution can be obtained from the solution for detuning one arbitrary oscillator ($x=b$) by superposition (subtraction) of two solutions, one for $b=a$ and one for $b=-a$. The time domain result is as shown.

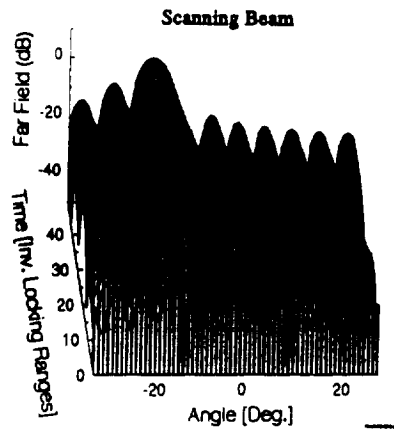
Beamsteering Phase



12

This is a graphical representation of the beamsteering phase solution just obtained.

Far Zone Radiation Pattern



13

This plot shows the dynamics of the far zone radiation pattern during beamsteering. It was obtained by computing the radiation pattern for each time value by integration over the aperture using the phase solution represented on the previous vugraph. Note that the beam integrity and sidelobe structure is maintained throughout the transient period.

Scanning via Injection

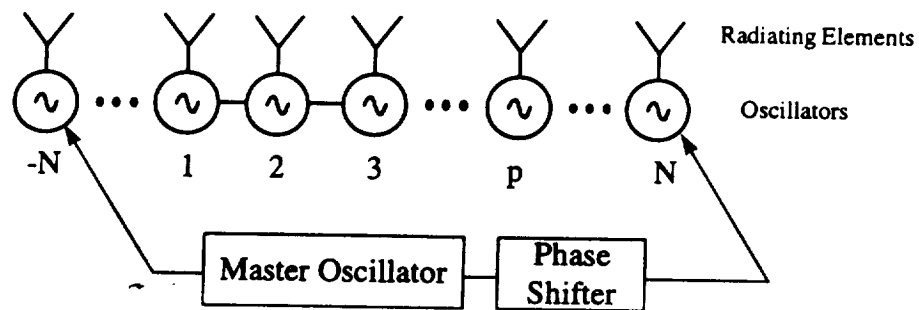
- Concept due to K. Stephan [IEEE Trans. MTT-34, pp. 1017-1025, October 1986].
 - Linear array of mutually injection locked VCOs.
 - External injection locking of end oscillators.
 - Shift relative phase of injection signals.
 - Linear aperture phase with variable gradient.
 - Analysis via numerical solution of a system of first order nonlinear differential equations based on Adler's theory of injection locking.

14

The fundamental concept of steering phased array beams by appropriately injection locking the end oscillators of a linear array originated with Karl Stephan circa 1986. He suggested that linear phase progressions along the array could be established if the end oscillators were injection locked to a common external source and a phase shifter were inserted in one line to control the relative phase of the two injection signals.

This analysis of the array took the form of numerical solution of a system of first order nonlinear differential equations derived using Adler's theory of injection locking. This made intuitive understanding of the dynamics difficult.

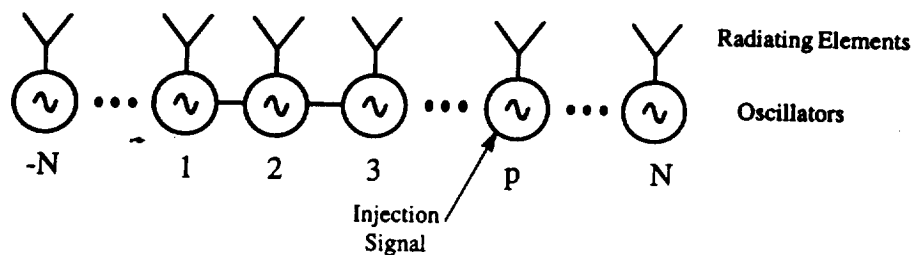
Stephan's Beamsteering Scheme



15

This diagram shows the Stephan scheme for beam steering. The master oscillator provides injection signals to the two end oscillators while the phase shifter controls the relative phase of these signals. The result is a linear phase progression across the array.

Injection Locked Coupled Oscillator Array



16

This shows the concept of controlling the phase in a radiating aperture using coupled electronic oscillators. The p^{th} oscillator is injection locked to an externally derived signal. It will be found that this signal provides a means of controlling the aperture phase distribution of the array.

Injection Model

Let the injection signal be represented by:

$$V(x) = \frac{\Delta\omega_{lock,p,inj}}{\Delta\omega_{lock}} \delta(x - p)$$

The continuum equation is,

$$\frac{\partial^2 \phi}{\partial x^2} - V(x)\phi - \frac{\partial \phi}{\partial \tau} = -\frac{\omega_{tune} - \omega_{ref}}{\Delta\omega_{lock}} - V(x)\phi_{inj}(\tau)$$

Laplace transformation yields,

$$\frac{\partial^2 f}{\partial x^2} - V(x)f - sf = -\frac{\omega_{tune} - \omega_{ref}}{\Delta\omega_{lock}} - V(x)f_{inj}(\tau)$$

17

We choose, for analytical convenience, to represent the injection term in terms of the Dirac delta function. Upon Laplace transformation with respect to the time variable, the differential equation finally takes the form shown. This differential equation forms the basis of the analysis to follow.

Finite Array Solution

Postulate, $\tilde{F}(x, s) = C_b e^{-|x-b|\sqrt{s}} + C_R e^{-x\sqrt{s}} + C_L e^{x\sqrt{s}} - \frac{C_0}{s^2}$

Boundary conditions at the injection points and Neumann conditions at the array ends results in,

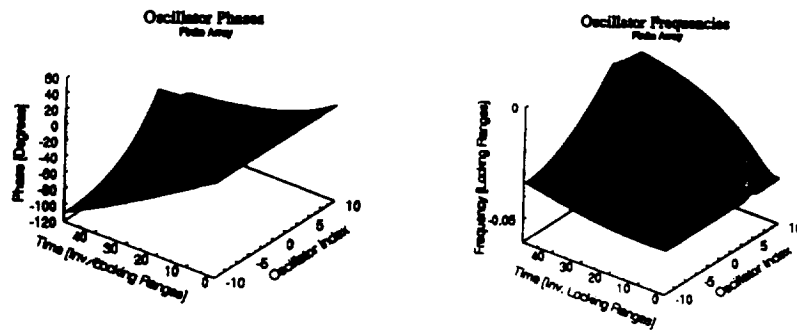
$$\tilde{F}(x, s) = \frac{C_0}{s^2} \left\{ \frac{C \cosh[\sqrt{s}(2a+1-|b-x|)] + C \cosh[\sqrt{s}(b+x)]}{2\sqrt{s} \sinh[\sqrt{s}(2a+1)] + C \cosh[\sqrt{s}(2b)] + C \cosh[\sqrt{s}(2a+1)]} \right\} - \frac{C_0}{s^2}$$

$$\tilde{W}(x, s) = \frac{\tilde{C}_0}{s} \left\{ \frac{C \cosh[\sqrt{s}(2a+1-|b-x|)] + C \cosh[\sqrt{s}(b+x)]}{2\sqrt{s} \sinh[\sqrt{s}(2a+1)] + C \cosh[\sqrt{s}(2b)] + C \cosh[\sqrt{s}(2a+1)]} \right\} - \frac{C_0}{s}$$

18

Postulating a solution of the form shown and applying the boundary conditions at the array ends and the slope discontinuity condition at the injection point to determine the unknown constants, we obtain the solutions shown for the phase and frequency distributions in the Laplace domain.

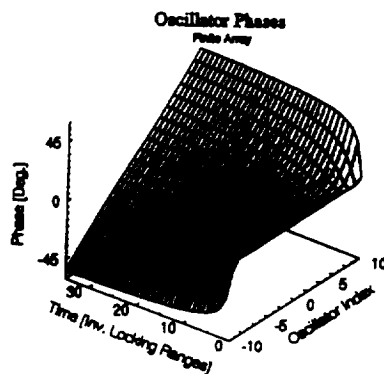
Finite Array with One Injection Point



19

The solution for the phase distribution of a twenty-one oscillator array with oscillator number five externally injection locked to a signal with a frequency step at time zero is shown on the left. The corresponding frequency solution is shown on the right.

Beam Steering Example



20

This is a graphical representation of the example of beam steering.

Gradual Phase Change

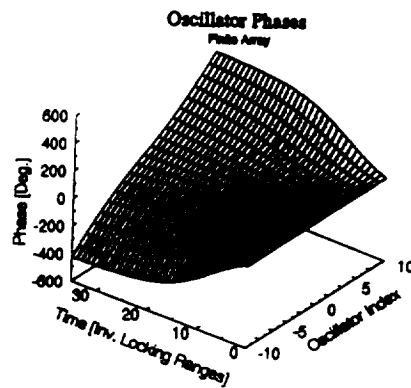
- Step injection phase change limited to less than ninety degrees.
 - Yields extremely limited beam steering angles.
 - Can be mitigated by gradual phase change.
- Gradual change result can be obtained by convolution with a Gaussian.
 - Time domain solution is expressed as a sum of exponentials.
 - Convolution of a Gaussian and an exponential can be expressed as multiplication by a function involving complementary error functions.

21

This solution has the disadvantage that the total phase difference between the array ends can be no more than 180 degrees leading to very small scan angles. This limit derives from the limitation on the phase difference between adjacent oscillators to 90 degrees. Fortunately, this can be mitigated by gradually changing the phase instead of stepping it. This allow the neighboring oscillator to follow the phase change and thus to minimize the difference. In that case the maximum end to end phase is 90 degrees time the number of oscillators less one.

Gradual Steering Example

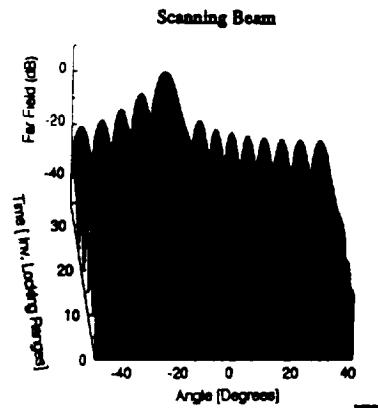
Choose, $\tau_0 = 6.0$
 $\alpha = 0.01$



22

The result is shown here and clearly provide much greater flexibility in terms of the phase excursion across the array which is obtainable; in this case nearly 1200 degrees.

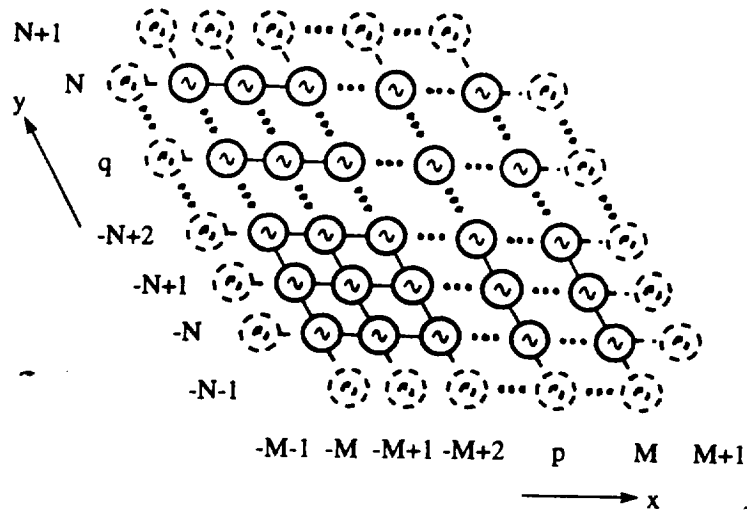
Far Zone Field



23

This is a plot of the far zone field pattern of an array of twenty-one elements separated by a half wavelength and fed with the signals from the twenty-one oscillator array with phase distribution shown in the previous vugraph. The beam steering dynamics is clearly displayed.

The M by N Array



24

This diagram schematically represents a $(2M+1)$ by $(2N+1)$ array of oscillators coupled to nearest neighbors. This is the array to be analyzed in the following. The oscillators shown in dashed lines are external sources which provide the properly phased injection signals to the perimeter oscillators of the array.

The Continuum Model

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - V(x, y)\phi - \frac{\partial \phi}{\partial \tau} = -\frac{\omega_{\text{tune}} - \omega_{\text{ref}}}{\Delta\omega_{\text{lock}}} - V(x, y)\phi_{\text{inj}}(x, y; \tau)$$

where,

$$\tau = \Delta\omega_{\text{lock}} t$$

25

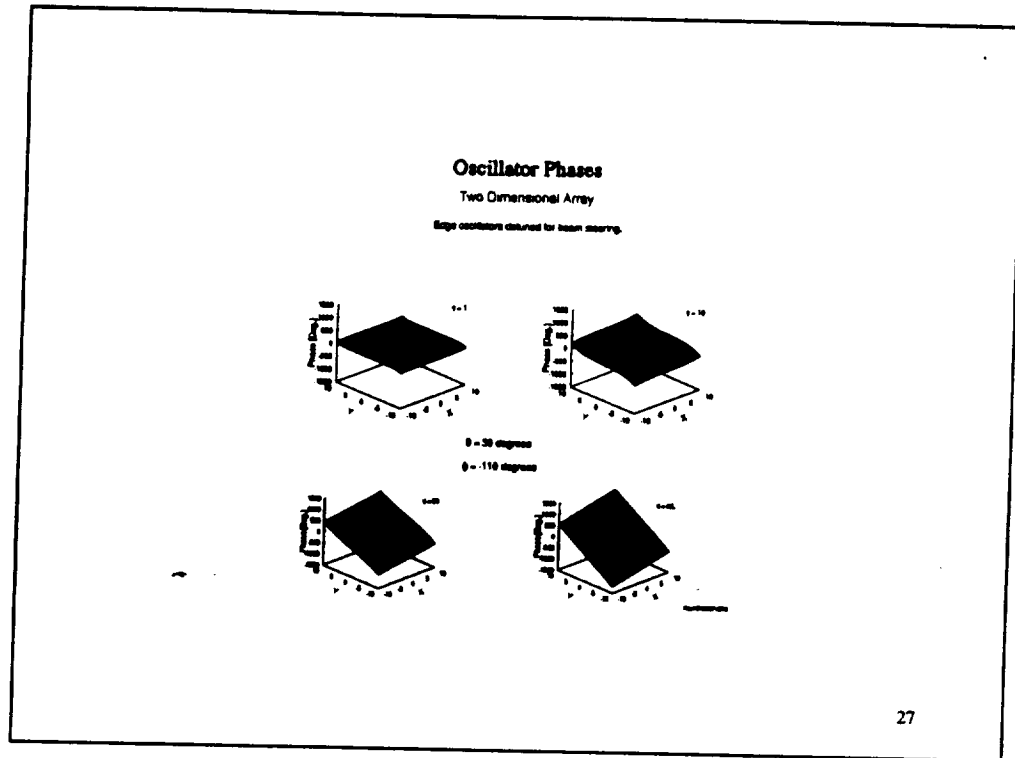
Thus, defining a continuous phi function and continuous variables x and y indexing the oscillators, we arrive at the partial differential equation for phi shown. As in the one dimensional case, V represents the distribution and strength of the injection signals with phase ϕ_{inj} . Tau is time measured in inverse locking ranges.

A Numerical Example

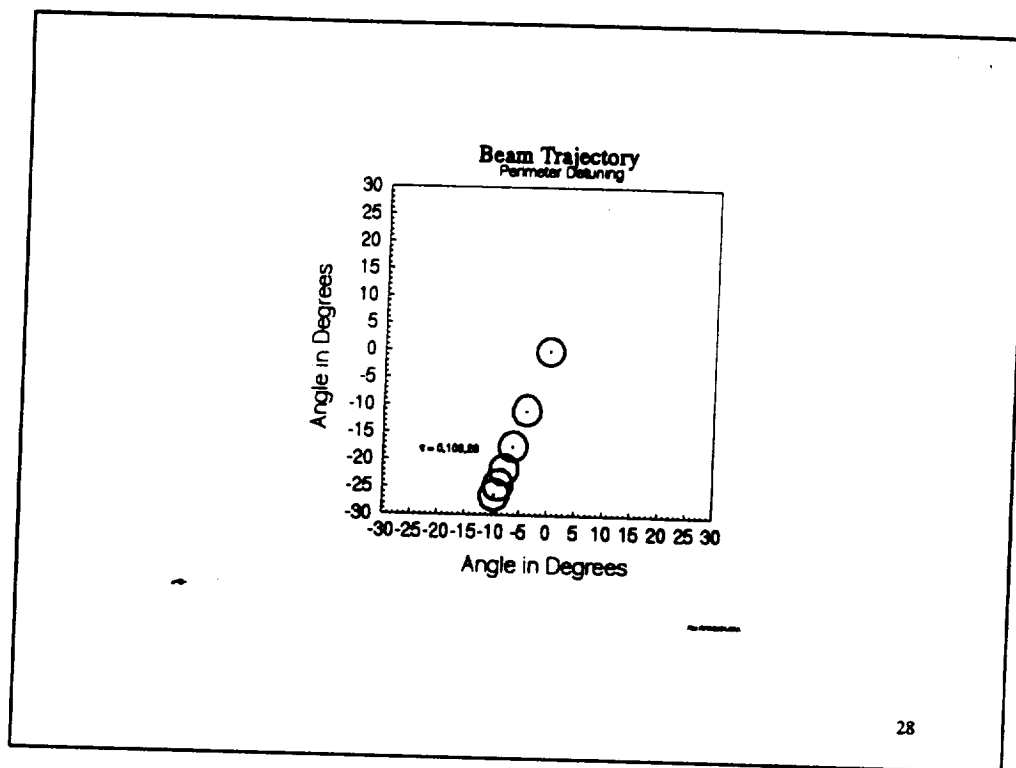
- Consider a 21 by 21 element square array.
- Radiating elements:
 - Half wavelength spacing
 - Connected to each oscillator

26

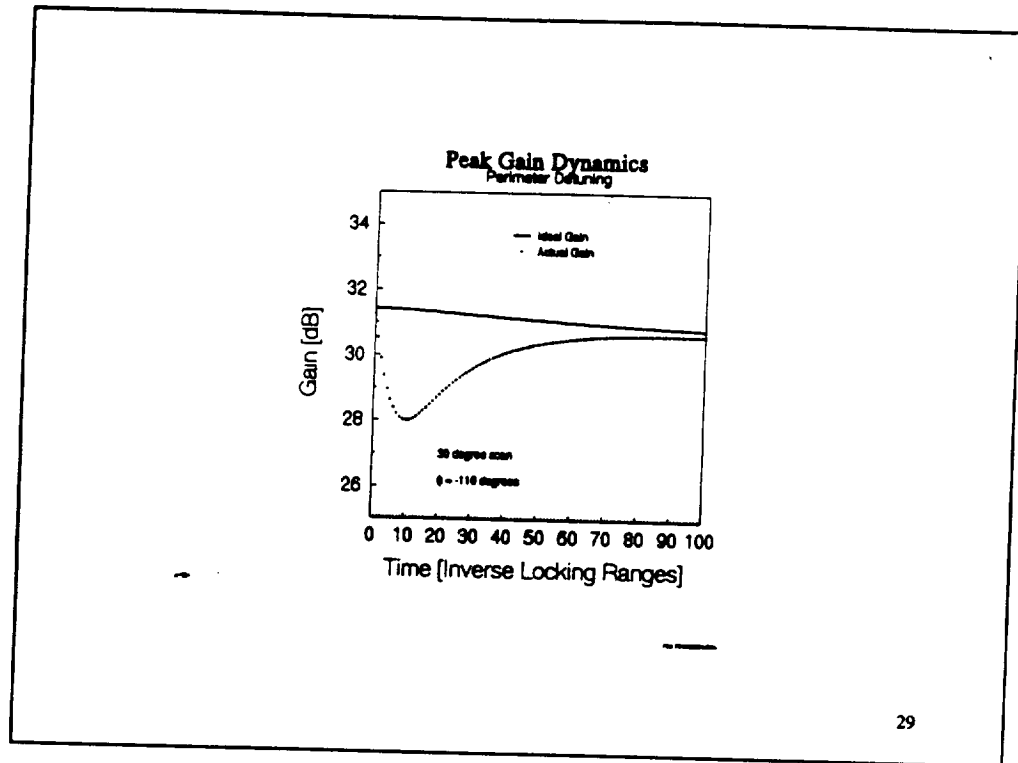
Consider a 21 by 21 element array with one radiating element connected to each oscillator. Let the radiating elements be spaced one half wavelength apart and let the external injection signals be applied to the perimeter oscillators per the preceding theory. The following vugraphs show a series of computed results concerning the aperture phase and far zone field of such an array.



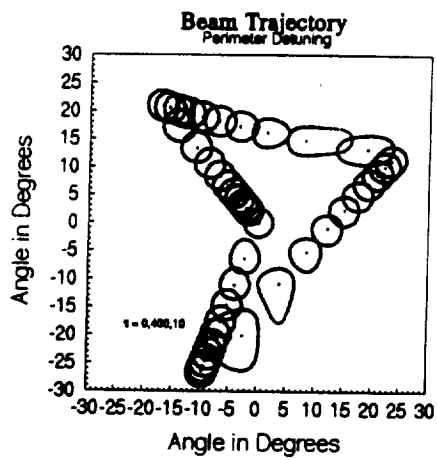
These graphs show the time evolution of the phase when detuning appropriate to beamsteering is applied. Note that the steering voltages are constant along each edge of the array.



This graph shows the beam peak (dots) and the three dB contour (closed curves) as a function of time during the beamsteering transient resulting when a step steering voltage designed to steer the beam thirty degrees off normal is applied at time zero.

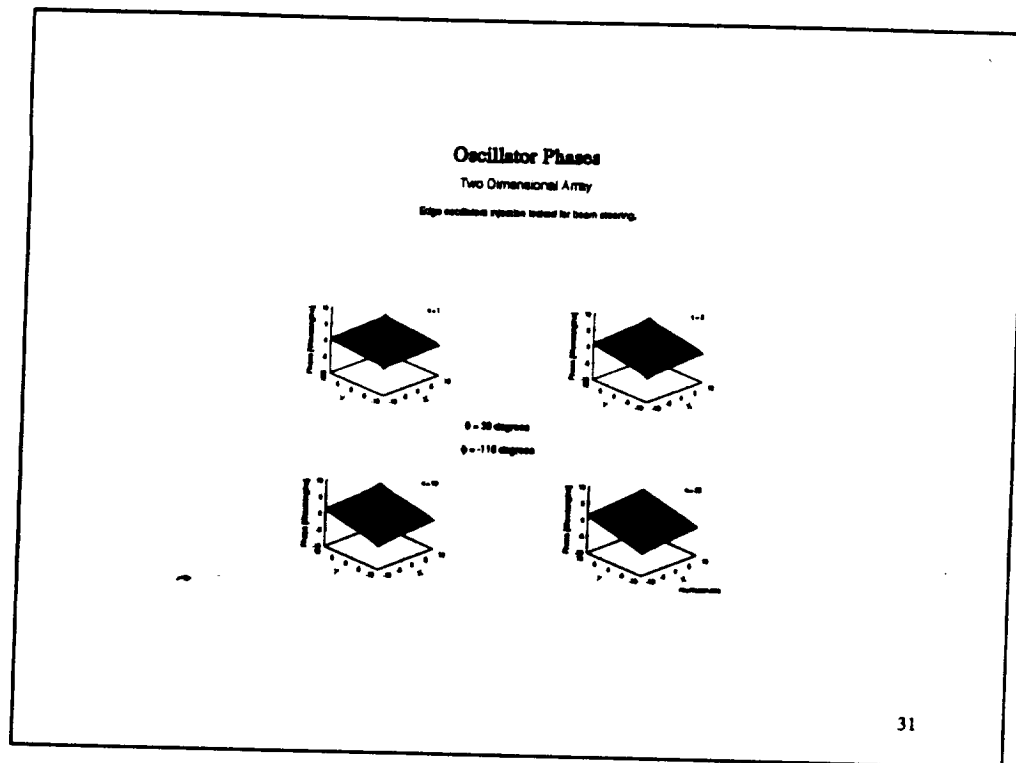


During the transient period, the aperture phase is nonplanar. This results in a temporary reduction in gain due to phase aberration. This graph shows this gain reduction as a function of time compared with the projected aperture loss to be expected for each beam position. These curves were obtained by pattern integration.

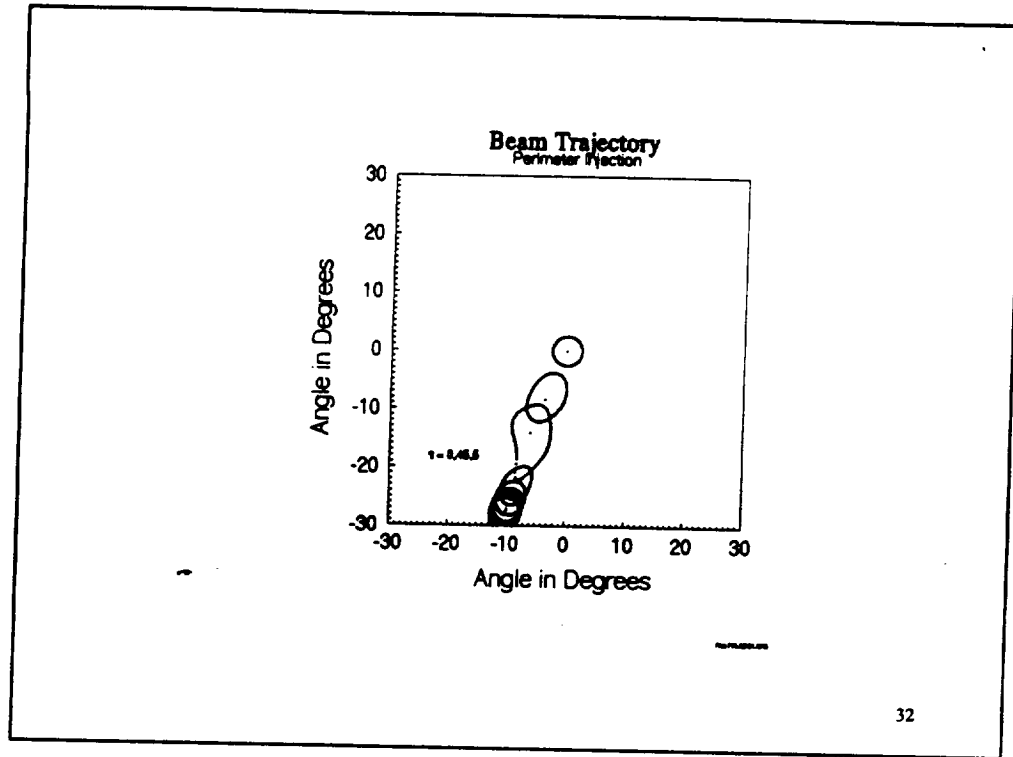


30

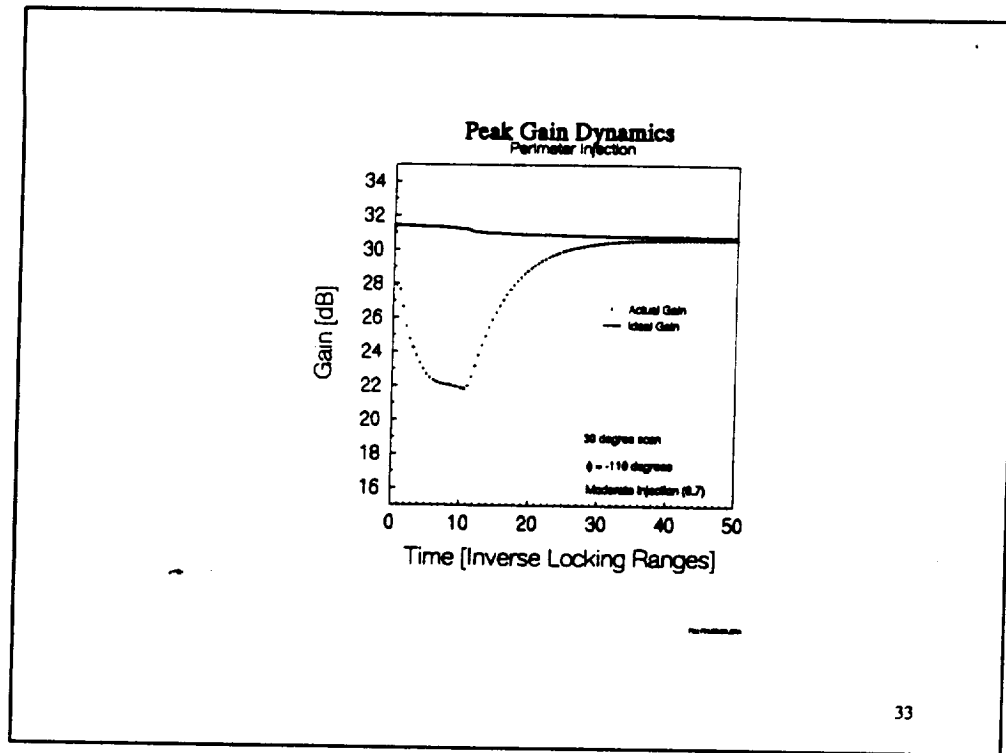
This graph shows the result of four sets of steering voltages applied in rapid succession. Note that the aberration effects seem to be greater when steering from one off axis position to another than when steering to or from normal.



These graphs show the time evolution of the phase when injection phase appropriate to beamsteering is applied. Note that this phase progression is linear along each edge of the array.

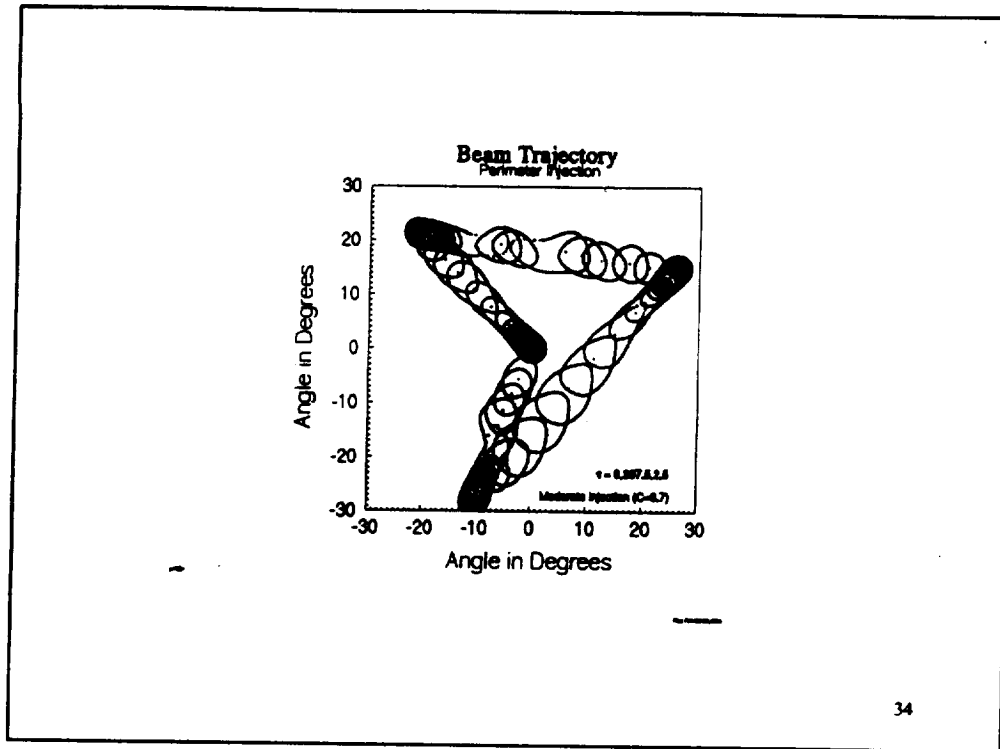


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During the transient period, the aperture phase is nonplanar. This results in a temporary reduction in gain due to phase aberration. This graph shows this gain reduction as a function of time compared with the projected aperture loss to be expected for each beam position. These curves were obtained by pattern integration.

The irregular behavior is attributed to the fact that the abscissa is time as opposed to angle. Thus, the irregularities are due to changes in the rate of beam motion.



This graph shows the result of four sets of steering injection phases applied in rapid succession. Note that the aberration effects seem to be greater when steering from one off axis position to another than when steering to or from normal.

Summary of Key Results

- Inter-oscillator phase difference
 - Limited to 90 degrees.
 - Limit can be mitigated by:
 - Reducing the element spacing.
 - Adding oscillators between the radiating ones.
 - Radiating at a harmonic of the coupling frequency.
- The response time of the array varies as the *square* of its length.
- The maximum step detuning of a single oscillator of a one dimensional array approaches *two locking ranges* for a large array.
- If all of the oscillators are injection locked the response time is that of a single oscillator.
- This linearized model substantiates the beam steering results of Stephan.
- The injection signal phase is limited to 90 degrees from injected oscillator phase unless applied gradually.
- The maximum step change in frequency of injection of one of the oscillators of a one dimensional array is limited to the locking range *divided by the number of oscillators*.

35

In conclusion, the chart summarizes the key results obtained to date via continuum modeling concerning the dynamic behavior of one and two dimensional coupled oscillator arrays involving either detuning or external injection locking of the oscillators.